Pin-Force and Euler–Bernoulli Models for Analysis of Intelligent Structures

Giulio Strambi,* Renato Barboni,† and Paolo Gaudenzi‡
University of Rome "La Sapienza," 00184 Rome, Italy

Nomenclature

k = curvature of the beam
t_b = thickness of the beam
z = out-of-plate coordinate
ϵ = normal strain in the longitudinal direction
Λ_{ai} = actuation strain

Introduction

In recent years several models have been developed for the prediction of the induced deformation produced by layered actuators on a simple beam-like structure.[1-3] In the present Note the pin-force and the Euler–Bernoulli models are formulated in a general fashion, and the relationship between them is discussed, especially for the case of one side only actuation layer. For both models the static response of the system is obtained for different geometrical, elastic, and piezoelectric characteristics in the cases of one and two (top and bottom surface bonded) actuation layers.

Pin-Force Model

The pin-force model of Crawley and de Luis[1] assumes that the deformation is transferred from the piezo to the substrate through actuation forces concentrated at the ends of the actuator patch. The actuator is modeled as a beam with only axial stiffness, whereas the substructure is modeled as an Euler–Bernoulli beam, with axial and bending stiffness. The governing equations are here derived by means of the principle of virtual work in the virtual forces version; the conditions of compatibility in correspondance of the pins are in this case fullfilled.

If $\delta\mathcal{L}_{\text{ext}}$ is the virtual work of the external forces, $\delta\mathcal{L}_b$ is the internal virtual work of the beam, $\delta\mathcal{L}_{a1}$ and $\delta\mathcal{L}_{a2}$ are the internal virtual works of the upper and lower actuation layers, respectively, then the principle of virtual work can be written as follows:

$$\delta\mathcal{L}_{\text{int}} = \delta\mathcal{L}_{a1} + \delta\mathcal{L}_{a2} + \delta\mathcal{L}_b = \delta\mathcal{L}_{\text{ext}}$$

The actuation strains Λ_{ai} as well as the actuation forces F_{ai} are decomposed in a symmetrical (extensional actuation Λ, F) and an antisymmetrical (flexural actuation $\tilde{\Lambda}$, $\tilde{F}$) part (Fig. 1):

$$\Lambda_{a1} = \Lambda + \tilde{\Lambda} \qquad \Lambda_{a2} = \Lambda - \tilde{\Lambda}$$

$$F_{a1} = F + \tilde{F} \qquad F_{a2} = F - \tilde{F}$$

where $F_{ai} = \sigma_{ai}\, A_{ai}$.

The constitutive relations are as follows. For the beam structure

$$\epsilon_b = \epsilon_0 + kz = \frac{\sigma_b}{E_b} = \frac{2F}{(EA)_b} + \frac{\tilde{F}t_b}{(EI)_b}\, z$$

and for the piezoelectric actuator

$$\epsilon_{ai} = \epsilon_{0ai} = \frac{\sigma_{ai}}{E_{ai}} + \Lambda_{ai} = \frac{F_{ai}}{(EA)_{ai}} + \Lambda_{ai}$$

where $i = 1, 2$.

Received Dec. 13, 1994; revision received April 11, 1995; accepted for publication April 11, 1995.

*Graduate Student, Dipartimento Aerospaziale, Via Eudossiana 16.
†Professor, Dipartimento Aerospaziale, Via Eudossiana 16.
‡Assistant Professor, Dipartimento Aerospaziale, Via Eudossiana 16.

By applying two virtual self-equilibrating force systems (δF^e is the pure extensional virtual force system and δF^f is the pure flexural virtual force system), two independent equations are obtained from which it is possible to calculate the two actual actuation forces, in absence of external loads,

$$\delta\mathcal{L}_{\text{int}} = \int_{V_{a1}+V_{a2}+V_b} \delta\sigma\, \epsilon\, \mathrm{d}V = \delta\mathcal{L}_{\text{ext}} = 0$$

Using the pure extensional virtual force-system δF^e,

$$\delta\mathcal{L}_{ai} = \int_{V_{ai}} \delta\sigma_{ai}^e\, \epsilon_{ai}\, \mathrm{d}V_{ai} = l\,(-\delta F^e)\left[\Lambda \pm \tilde{\Lambda} - \frac{F}{(EA)_{ai}} \mp \frac{\tilde{F}}{(EA)_{ai}}\right]$$

$$\delta\mathcal{L}_b = \int_{V_b} \delta\sigma_b^e\, \epsilon_b\, \mathrm{d}V_b = l\,(2\delta F^e)\left[\frac{2F}{(EA)_b}\right]$$

the following equation is derived:

$$(4 + \psi_1 + \psi_2)\, F + (\psi_1 - \psi_2)\, \tilde{F} = 2\,\Lambda\,(EA)_b$$

where $\psi_i = (EA)_b/(EA)_{ai}$.

Using the pure flexural virtual force-system δF^f,

$$\delta\mathcal{L}_{ai} = \int_{V_{ai}} \delta\sigma_{ai}^f\, \epsilon_{ai}\, \mathrm{d}V_{ai} = l\,(\mp\delta F^f)\left[\Lambda \pm \tilde{\Lambda} - \frac{F}{(EA)_{ai}} \mp \frac{\tilde{F}}{(EA)_{ai}}\right]$$

$$\delta\mathcal{L}_b = \int_{V_b} \delta\sigma_b^f\, \epsilon_b\, \mathrm{d}V_b = l\,\delta F^f\, t_b\left[\frac{\tilde{F}t_b}{(EI)_b}\right]$$

the second equation is thus obtained:

$$(\psi_1 - \psi_2)\, F + (12\rho + \psi_1 + \psi_2)\, \tilde{F} = 2\,\tilde{\Lambda}\,(EA)_b$$

where $\rho = A_b t_b^2/12 I_b$. In this way the following system has to be solved:

$$\begin{vmatrix} \psi_1 + \psi_2 + 4 & \psi_1 - \psi_2 \\ \psi_1 - \psi_2 & \psi_1 + \psi_2 + 12\rho \end{vmatrix} \begin{vmatrix} F \\ \tilde{F} \end{vmatrix} = 2\,(EA)_b \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} \Lambda \\ \tilde{\Lambda} \end{vmatrix}$$

The solution has the form

$$\begin{vmatrix} F \\ \tilde{F} \end{vmatrix} = \frac{2(EA)_b}{\Delta} \begin{vmatrix} \psi_1 + \psi_2 + 12\rho & \psi_2 - \psi_1 \\ \psi_2 - \psi_1 & \psi_1 + \psi_2 + 4 \end{vmatrix} \begin{vmatrix} \Lambda \\ \tilde{\Lambda} \end{vmatrix}$$

where $\Delta = 4[(\psi_1 + \psi_2)(1 + 3\rho) + \psi_1\psi_2 + 12\rho]$.

From the expression of the actuation forces is possible to derive the expressions of the normal strain and of the curvature

$$\epsilon_0 = 2F/(EA)_b \qquad k = \tilde{F}t_b/(EI)_b$$

For rectangular sections of unitary width

$$\rho = 1 \qquad \psi_i = \bar{E}_i T_i$$

where $\bar{E}_i = E_b/E_{ai}$, and $T_i = t_b/t_{ai}$.

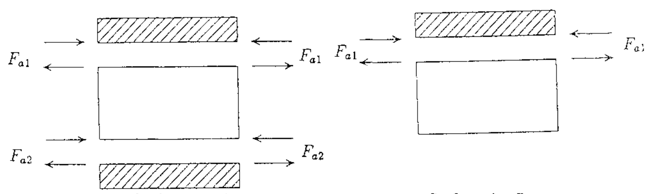

Fig. 1 Configuration A, two piezoelectric layers bonded on top and bottom surfaces; and configuration B, one piezoelectric layer bonded on one surface only.

Three typical response problems, referred to two characteristic configurations, are now examined. The first two response problems have two symmetric actuator patches $\psi_1 = \psi_2 = \psi$. The first of these is activated in phase $\Lambda_{a1} = \Lambda_{a2} = \Lambda$, $\tilde{\Lambda} = 0$

$$F = \frac{(EA)_b}{2+\psi} \Lambda_{a1}$$

$$\epsilon_0 = \frac{2}{2+\psi} \Lambda_{a1}$$

The second is activated with opposite phase $\Lambda_{a1} = -\Lambda_{a2} = \tilde{\Lambda}$, $\Lambda = 0$

$$\tilde{F} = \frac{(EA)_b}{6+\psi} \Lambda_{a1} = \frac{6}{6+\psi} \frac{2(EI)_b}{t_b^2} \Lambda_{a1}$$

$$k = \frac{6}{6+\psi} \frac{2}{t_b} \Lambda_{a1}$$

The third response problem has a one-sided actuator patch (for example, actuator 1, $1/\psi_2 = 0$, and $\Lambda_{a2} = 0$, $F_{a2} = 0$) and we obtain

$$F = \tilde{F} = \frac{(EA)_b}{2\,(4+\psi_1)} (\Lambda + \tilde{\Lambda})$$

$$F_{a1} = 2F = \frac{(EA)_b}{4+\psi_1} \Lambda_{a1}$$

$$\epsilon_0 = [1/(4+\psi_1)]\,\Lambda_{a1} \tag{1}$$

$$k = [3/(4+\psi_1)](2/t_b)\,\Lambda_{a1} \tag{2}$$

Euler–Bernoulli Model

The Euler–Bernoulli model assumes the plane section hypothesis along the whole thickness, including the actuators that contribute to the bending stiffness of the overall structure.

Constitutive Relations

Plate structure:

$$\sigma_b = E_b\,\epsilon = E_b(\epsilon_0 + kz)$$

Piezoelectric layer:

$$\sigma_{ai} = E_{ai}(\epsilon - \Lambda_{ai}) = E_{ai}(\epsilon_0 + kz - \Lambda_{ai})$$

Equilibrium Equations

To obtain the governing equations of the problem, the principle of the virtual work, in the virtual displacement version, is used. In absence of external applied loads the virtual internal load $\delta\mathcal{L}_{\text{int}}$ is equal to zero:

$$\delta\mathcal{L}_{\text{int}} = \int_{V_{a1}+V_{a2}+V_b} \sigma\,\delta\epsilon\,\mathrm{d}V = \int_0^l \int_{-\frac{t_b}{2}-t_a}^{\frac{t_b}{2}+t_a} \sigma\,\delta\epsilon\,\mathrm{d}z\,\mathrm{d}x = 0$$

Following the classical techniques it is easy to obtain the following set of linear algebraic equations:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} \begin{vmatrix} \epsilon_0 \\ k \end{vmatrix} = \begin{vmatrix} (EA)_{a1}\Lambda_{a1} + (EA)_{a2}\Lambda_{a2} \\ (EA)_{a1}h_{a1}\Lambda_{a1} - (EA)_{a2}h_{a2}\Lambda_{a2} \end{vmatrix}$$

where

$$h_{ai} = (t_b + t_{ai})/2$$

$$a_{11} = (EA)_b + (EA)_{a1} + (EA)_{a2}$$

$$a_{12} = (EA)_{a1}h_{a1} - (EA)_{a2}h_{a2}$$

$$a_{22} = (EI)_b + (EI)_{a1} + (EI)_{a2} + (EA)_{a1}h_{a1}^2 + (EA)_{a2}h_{a2}^2$$

By inverting the system, is possible to write the expressions for ϵ_0 and k. As particular cases the solutions for the two typical configurations discussed in the preceding section (one- and two-sided actuators) are derived, with the assumption of rectangular sections.

With two symmetric actuator patches

$$E_{a1} = E_{a2}, \quad t_{a1} = t_{a2}, \quad h_{a1} = h_{a2}$$

When activated in phase $\Lambda_{a1} = \Lambda_{a2} = \Lambda$, $\tilde{\Lambda} = 0$

$$\epsilon_0 = \frac{2}{2+\psi} \Lambda_{a1}$$

$$k = 0$$

When activated out of phase $\Lambda_{a1} = -\Lambda_{a2} = \tilde{\Lambda}$, $\Lambda = 0$

$$\epsilon_0 = 0$$

$$k = \frac{6[1+(1/T)]}{6+\psi+(12/T)+(8/T^2)} \frac{2}{t_b} \Lambda_{a1}$$

where $T = t_b/t_a$.

With one-sided actuator patch (actuator 1)

$$t_{a2} = 0, \quad \Lambda_{a2} = 0$$

$$\epsilon_0 = \frac{1 + \dfrac{1}{\psi_1 T^2}}{4 + \psi_1 + \dfrac{6}{T} + \dfrac{1}{\psi_1 T^2} + \dfrac{4}{T^2}} \Lambda_{a1} \tag{3}$$

$$k = \frac{3[1+(1/T)]}{4+\psi_1+(6/T)+(1/\psi_1 T^2)+(4/T^2)} \frac{2}{t_b} \Lambda_{a1} \tag{4}$$

Discussion

In the preceding expressions is easy to recognize some terms that for high values of the parameter T (for aluminum–PZT systems, $T \geq 8$) become negligible, and the expression containing the remaining terms is the one obtained in the pin-force model, where the actuators have only axial but not bending stiffness. In fact, for high values of T (the beam is thicker than the piezo) and/or of $\psi_1 = \bar{E}T$ [$\bar{E} = E_b/E_a$, for a given T, the beam is stiffer than the piezo ($E_b > E_a$)], in the Euler–Bernoulli model the contribution of the actuators bending stiffness to the global stiffness of the system is negligible. This way the difference between the solutions of the two models vanishes as T and/or ψ_1 increase.

The solution obtained for one-sided actuator shows how the system is excited by a combined bending-compressive actuation, that is, the actuator induces a combined flexural-extensional deformation. In Fig. 2 the normalized curvature predicted by the pin-force model here developed, Eq. (2), is plotted as a function of the thickness ratio T for $\bar{E} = 5$, 1, and 0.2 and compared with Euler–Bernoulli results, Eq. (4). By comparing the diagrams obtained at a fixed value of the $\bar{E}$ ratio, it can be seen that it is possible to establish a value of T above which the predictions obtained with the Euler–Bernoulli and the pin-force model are practically the same. From Fig. 2 it can also be noted that this value is not fixed but depends on the ratio $\bar{E}$; the higher is Young's modulus of the material of the beam with respect to Young's modulus of the actuator, the lower is T.

In Fig. 3 the diagrams of the normalized longitudinal strain predicted by Eqs. (1) and (3) are plotted. From Eqs. (1) and (2) it is possible to obtain the value of the deformation at the actuator-substructure interface,

$$\epsilon\,|_{z=t_b/2} = \epsilon_0 + k\,\frac{t_b}{2} = \frac{4}{4+\psi} \Lambda_{a1}$$

It is possible to introduce a parameter α, as done in Crawley and

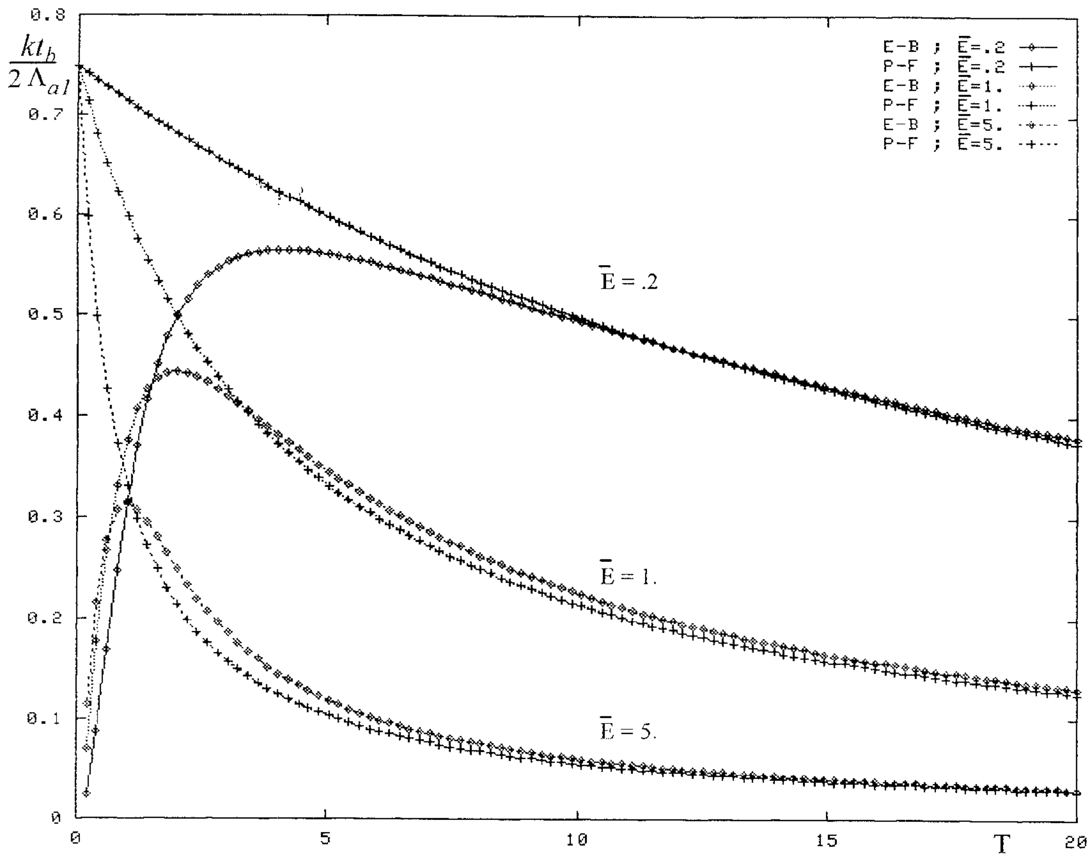

Fig. 2 Normalized curvature for an actuator patch on one side only, $\bar{E} = E_b/E_a$.

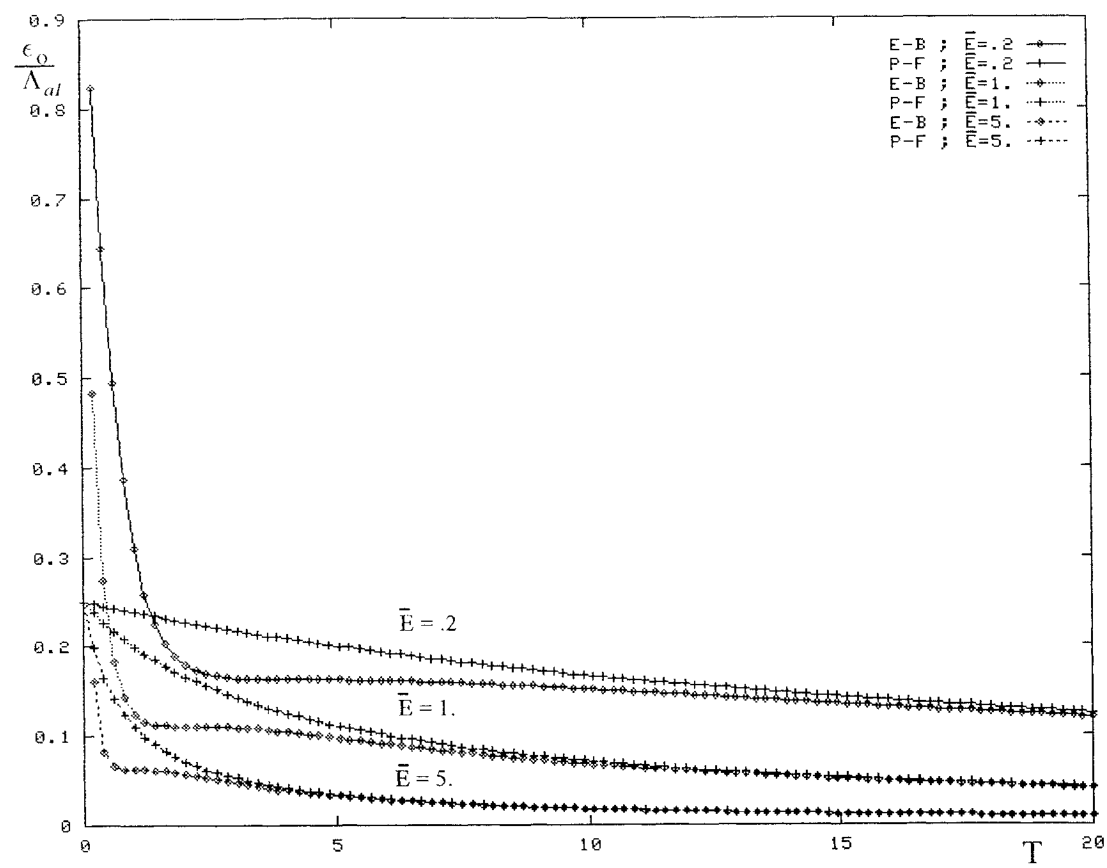

Fig. 3 Normalized longitudinal strain for an actuator patch on one side only, $\bar{E} = E_b/E_a$.

de Luis,[1] to write a simple expression of the axial strain at the interface, valid for high values of T,

$$\epsilon\,|_{z=t_b/2} = \frac{\alpha}{\alpha+\psi}\,\Lambda_{a1}$$

In the case of symmetric actuator patches, $\alpha = 2$ for pure extension and $\alpha = 6$ for pure bending; for an actuator on one side only $\alpha = 4$, as shown before. In the present formulation the fully extensional-bending coupling was accounted for; in fact, the axial deformation is always present in the case of one-sided actuator.

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Computation of Eigenvector Derivatives with Repeated Eigenvalues Using a Complete Modal Space

De-Wen Zhang*
Beijing Institute of Structure and Environment, 10076 Beijing, People's Republic of China
and
Fu-Shang Wei†
Kaman Aerospace Corporation, Bloomfield, Connecticut 06002

Nomenclature

$\bar{H}_h, H_h$ = equivalent high-order modal matrix
K = real symmetric stiffness matrix
$K_{,j}$ = derivative of stiffness matrix with respect to p_j
$K_{,jj}$ = second-order derivative of stiffness matrix
M = real symmetric mass matrix
$M_{,j}$ = derivative of mass matrix with respect to p_j
$M_{,jj}$ = second-order derivative of mass matrix
$Z_{,j}$ = derivative matrix of eigenvector of repeated root with respect to design parameter p_j
Λ = complete eigenvalue diagonal matrix
$\bar{\Lambda}$ = repeated eigenvalue diagonal matrix
$\bar{\Lambda}_{,j}$ = derivative diagonal matrix of repeated eigenvalue with respect to p_j
$\bar{\Lambda}_{,jj}$ = diagonal matrix of second-order derivative of repeated eigenvalue
$\bar{\lambda}$ = repeated eigenvalue
Φ = complete eigenvector matrix
Φ_h, Λ_h = high-order eigenpair matrix
Φ_ℓ, Λ_ℓ = lower order eigenpair matrix
$\bar{\Phi}$ = equivalent complete modal matrix
φ = eigenvector (column vector)
Ψ, Z = eigenvector matrix of repeated eigenvalue

Received March 30, 1992; revision received Feb. 23, 1995; accepted for publication Feb. 23, 1995.

*Senior Research Engineer. Member AIAA.

†Principal Engineer, Engineering and Development Department. Member AIAA.

I. Introduction

Eigenvector derivatives are important for structural dynamics, optimal analysis, and control system design. Recently, eigenvector derivative data have become necessary for use in structural model modification,[1–4] system parameter identification,[5–6] and failure diagnosis.[7] In addition, the calculation of eigenvector derivatives has an important role in sensitivity analysis. At present, we have several procedures[8–19] for computing eigenvector derivatives. The governing equation of the eigenvector derivative is defined as

$$(K - \lambda_i M)\varphi_{i,j} = (\lambda_i M_{,j} + \lambda_{i,j} M - K_{,j})\varphi_i = g_i \tag{1}$$

The coefficient matrix $(K - \lambda_i M)$ is degenerate for m repeated eigenvalues and has rank n–m. In Refs. 8 and 9, the authors solved Eq. (1) by adding one or m independent equations to Eq. (1) through the relation of mass orthogonality. In Refs. 10–15 the authors introduced a procedure to divide the solution technique into two steps: first solve the particular solution by exerting one or m given restrictions on Eq. (1), then find the constants existing in the general solution through the relation of mass orthogonality. In Ref. 15 the author pointed out that the procedure for solving the particular solution in Refs. 11–13 may fail under certain circumstances. Employing the more rigorous method in Ref. 14, this failure can be avoided. In Refs. 16 and 17 the particular solution is yielded by using the matrix disturbance method. In comparison with Nelson method[10] and extended Nelson methods,[11–14] the direct perturbed (DP) method described in Refs. 16 and 17 has the advantages of numerical simplicity and efficiency for solving the particular solution and possibly makes the particular solution approach the general solution directly.

The modal method used in Ref. 8 avoided directly solving Eq. (1). The modal method presented in Ref. 8 was based on the incomplete modal space $\varphi_k (k = 1, 2, \ldots, \ell < n)$. The eigenvector derivative is expressed as

$$\varphi_{i,j} = \sum_{k=1}^{\ell} q_k \varphi_k \tag{2}$$

where ℓ is the number of lower order modes, and n is the total number of degrees of freedom of the analytic model. The precision of the eigenvector derivatives using the incomplete modes φ_k is poor. To reduce the modal truncation error, the author of Ref. 18 proposed an approach to add a static solution to the incomplete modes to improve the solution,

$$\varphi_{i,j} = \varphi_{i,j}^{(0)} + \sum_{k=1}^{\ell} q_k \varphi_k \tag{3}$$

where $\varphi_{i,j}^{(0)}$ can be obtained from Eq. (1) with λ_i equal to zero as shown in Eq. (4),

$$K\varphi_{i,j}^{(0)} = g_i \tag{4}$$

when K is singular, we utilize the "dynamic solution" method for $\varphi_{i,j}^{(0)}$ as proposed in Refs. 16 and 17 and shown as

$$[K - (1+\varepsilon)\lambda_i M]\varphi_{i,j}^{(0)} = g_i \tag{5}$$

in which ε is a small perturbation parameter. The application of the incomplete modal expansion method with static correction described by Eqs. (3) and (4) to the calculation of eigenvector derivatives with repeated eigenvalues is presented in the Appendix. The development of the incomplete modal expansion methods was described in Ref. 19.

This paper presents a complete modal space (CMS) method that is an exact modal expansion method. There is no modal truncation error existing in the derivation. Also, there is no additional effort required to find $\varphi_{i,j}^{(0)}$ as shown in Eq. (3). Moreover, CMS can be used in the case of repeated eigenvalues. As already mentioned, this method gives better results compared to those obtained from Ref. 8 and Eq. (3).